

Geometrization of the Lienard-Wiechert Field

M. Sorg

Institut für Theoretische Physik der Universität Stuttgart

Z. Naturforsch. **37a**, 201–207 (1982); received August 21, 1981

A geometric description of the Lienard-Wiechert field of a single point charge is given. The dual of the Lienard-Wiechert field can be considered as the curvature in a bundle of 2-planes distributed smoothly over space-time. This geometric construction also accounts for the Teitelboim splitting of the total field into bound and radiative parts: The radiation field can be identified with the curvature in the normal bundle. Various questions concerning potentials and the integrability of the distributions involved are discussed.

Whereas the intrinsic symmetry group for the field is that of the 2-plane bundle, namely $SO(2)$, the normal bundle approach leads to the group $SO(1,1)$ as the internal symmetry group for the radiation field.

Introduction

Recently, a purely geometric description has been given for point particles a sources for the electromagnetic field, as well non-relativistically [1, 2] as in the relativistic case [3]. The essential point in such a geometric description is that the physical concept of the electromagnetic field now arises as a geometric notion, namely as the curvature field in an appropriate fibre bundle over space-time as base manifold. In this approach, it has been turned out that the electromagnetic 2-form can be considered as the intrinsic curvature of a 2-dimensional, real, oriented, unit vector bundle, a construction which is known as Euler class in differential topology [4].

The associated bundle of orthonormal frames is defined (locally) by two space-like vector fields $\mathbf{k}(x)$ and $\mathbf{h}(x)$

$$(\mathbf{k} \cdot \mathbf{k}) = (\mathbf{h} \cdot \mathbf{h}) = -1, \quad (\mathbf{k} \cdot \mathbf{h}) = 0,$$

and the curvature field in question can then be expressed as

$$*F_{\mu\nu} = (\mathbf{k}_{,\mu} \cdot \mathbf{h}_{,\nu}) - (\mathbf{k}_{,\nu} \cdot \mathbf{h}_{,\mu}). \quad (1)$$

It has been shown in [1–3] that any field of the form (1) is due to quantized point charges

$$\oint *F = 4\pi n, \quad n = 0, \pm 1, \pm 2, \dots \quad (2)$$

This approach might be called “topological charge quantization”.

Reprint requests to Dr. M. Sorg, Institut für Theoretische Physik, Universität Stuttgart, Pfaffenwaldring 57, D-7000 Stuttgart 80.

In the form (1), the invariance of the electromagnetic field under a local $SO(2)$ transformation

$$\begin{aligned} \mathbf{k}' &= \cos \alpha \cdot \mathbf{k} + \sin \alpha \cdot \mathbf{h}, \\ \mathbf{h}' &= -\sin \alpha \cdot \mathbf{k} + \cos \alpha \cdot \mathbf{h}, \end{aligned} \quad (3)$$

becomes manifest. However, the electromagnetic field (1) exhibits a further invariance, as was already pointed out in [3]. If one introduces the orthonormal vector fields $\mathbf{u}(x)$ ($\mathbf{u} \cdot \mathbf{u} = +1$) and $\mathbf{v}(x)$ ($\mathbf{v} \cdot \mathbf{v} = -1$) and uses the following decomposition of the Minowski metric

$$\begin{aligned} \{g_{\mu\nu}\} &= \text{diag}(1, -1, -1, -1) \\ g_{\mu\nu} &= u_\mu u_\nu - v_\mu v_\nu - k_\mu k_\nu - h_\mu h_\nu \end{aligned}$$

then the field (1) can be put into the alternative form

$$\begin{aligned} *F_{\mu\lambda} &= -\varepsilon_{\alpha\beta\gamma\delta} u^\alpha u^\gamma v^\beta v^\delta \\ &\quad \cdot (u^\alpha_{,\mu} u^\beta_{,\lambda} - v^\alpha_{,\mu} v^\beta_{,\lambda}). \end{aligned} \quad (4)$$

In this latter version, the invariance of $*F$ under the local $SO(1,1)$ transformation

$$\begin{aligned} \mathbf{u}' &= \cosh \beta \cdot \mathbf{u} + \sinh \beta \cdot \mathbf{v}, \\ \mathbf{v}' &= \sinh \beta \cdot \mathbf{u} + \cosh \beta \cdot \mathbf{v} \end{aligned} \quad (5)$$

is easily verified.

The main result of the present paper is now, that the emergence of the group $SO(1,1)$ is not merely an artificial construction but the group $SO(1,1)$ as the gauge group of the normal bundle is of its own relevance. This is shown in the single point particle case by identifying the curvature in the normal bundle as the radiation part ${}^r F$ of the total Lienard-Wiechert field ${}^{LW} F$ the dual of which is the curva-

0340-4811 / 82 / 0300-0201 \$ 01.30/0. — Please order a reprint rather than making your own copy.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition “no derivative works”). This is to allow reuse in the area of future scientific usage.

ture in the initial $SO(2)$ bundle. In this way, one arrives at the curious result that the proper gauge group for the radiation field is $SO(1,1)$ instead of $SO(2) \cong U(1)$, which everyone assumes to be the right symmetry group of Maxwell's electrodynamics and which is indeed the relevant group when considering the total electromagnetic field. Thus, the present model yields also a geometric approach to the well-known Teitelboim splitting in classical point particle electrodynamics [5], where the total field is split up into the bound and radiation part.

Our procedure to establish the mentioned facts is the following: In Sect. I, retarded coordinates are introduced in modified space-time M_4^- . The modification consists of removing the time-like world line $\mathcal{L}^{(1)}$ of the point particle from ordinary Minkowski space M_4 : $M_4^- = M_4 \setminus \mathcal{L}^{(1)}$. The retarded coordinates provide us with a special section $e(x)$ in the trivial bundle (\mathcal{A}_4^-) of orthonormal frames over modified space-time M_4^- .

In Sect. II, the trivial tangent bundle (τ_4^-) is decomposed into the Whitney sum of two non-trivial bundles

$$(\tau_4^-) = (\hat{\tau}_4) \oplus (\tilde{\tau}_4). \quad (6)$$

This decomposition is equivalent to a special reduction of the Lorentz group $SO(1,3) \rightarrow SO(1,1) \times SO(2)$. The connections $\hat{\omega}$ and $\tilde{\omega}$ in the reduced subbundles are computed by means of the projection technique [6]. The curvatures $\hat{\Omega}$ and $\tilde{\Omega}$ of the connections $\hat{\omega}$ and $\tilde{\omega}$ then turn out to be the radiation field $r\tilde{\mathcal{F}}$ and the dual $*\tilde{\mathcal{F}}$ of the Lienard-Wiechert field ${}^{LW}\tilde{\mathcal{F}}$ of an arbitrarily accelerated point charge.

We want to stress here the fact that the identification of the curvature field with a physical notion (particle field) causes some ambiguity. One could equally well identify the curvature field with the electromagnetic field of a magnetic monopole, which would imply the interchange of $\tilde{\mathcal{F}}$ and $*\tilde{\mathcal{F}}$, and the magnetic charge would become a topological invariant rather than the electric charge. This would be also in agreement with the usual procedure in gauge theories, where the curvature field of the bundle involved is put equal directly to the electromagnetic field $\tilde{\mathcal{F}}$. However, we are interested here in a geometrization of ordinary Maxwellian electrodynamics of electric point charges, where the existence of magnetic monopoles is usually excluded. Our deviation from the usual procedure with respect to the identification of

physical and geometric notions does not cause any trouble here because of the duality invariance of Maxwell's equations.

In Sect. III, we consider a special $SO(1,1)$ gauge transformation, which transforms the light-like radiation potential obtained in Sect. II into a well-known space-like expression. Based on this space-like form of the radiation potential, one can give a further geometric characterization of the radiation field (Sect. IV): It is the differential of the mean curvature 1-form of the integral surfaces with respect to that distribution which is constructed by means of the vector fields $k(x)$ and $h(x)$. Since the mean curvature 1-form does not belong to the intrinsic geometry of the original $SO(2)$ bundle, rather to its extrinsic geometry, this second approach to the radiation field is an additional hint that the radiation field cannot be fitted into a pure $SO(2)$ description of the electrodynamics of a single particle provided one works with the particular construction used in this paper. In view of this fact, the group $SO(1,1)$ receives an independent meaning within the present framework.

The paper concludes with a short discussion of possible generalizations of the special results found in this paper.

I. Retarded Coordinates in M_4^-

According to the philosophy described in the foregoing Section, a point particle is a hole in the physical space, and the subsequent position of a single hole traces out a time-like world line $\mathcal{L}^{(1)}$ in ordinary Minkowski space:

$$\mathcal{L}^{(1)}: x = z(s), \quad (dz \cdot dz) = : ds^2 > 0. \quad (7)$$

Removing the world line $\mathcal{L}^{(1)}$ from M_4 changes the topology of ordinary space-time, the modified manifold M_4^- becoming homeomorphic to $\mathbb{R}^2 \times S^2$. This is readily recognized by introducing a new coordinate system in M_4^- , which is adapted to this topology (s. Figure 1).

The world line $\mathcal{L}^{(1)}$ is taken as the starting point for the new coordinate system using proper time s of the world line as the first coordinate $s(x)$ of an arbitrary point $x \in M_4^-$. Here, $s(x)$ is that value of proper time (up to an additive constant), which is determined by the intersection $z(x)$ of $\mathcal{L}^{(1)}$ and the backward light cone $l^-(x)$ with vertex in x :

$$z(x) = \mathcal{L}^{(1)} \cap l^-(x). \quad (8)$$

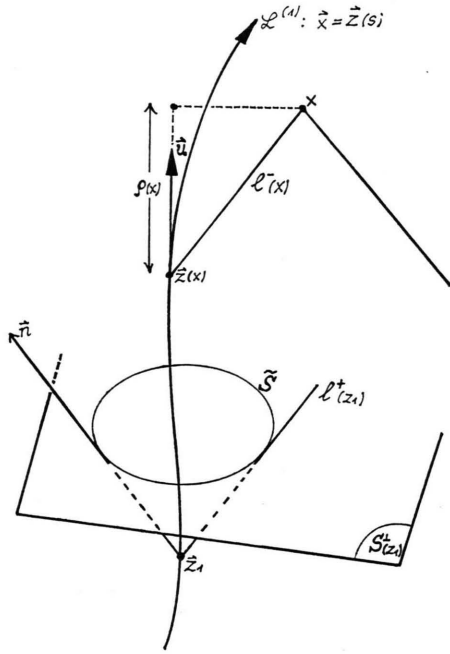


Fig. 1. The retarded construction. The retarded distance $\varrho(x)$ is the scalar product of the retarded velocity \mathbf{u} and the light-like separation vector $(\mathbf{x} - \mathbf{z}(x))$ between the field point x and the advanced particle position $\mathbf{z}(x)$:

$$\varrho(x) = (\mathbf{x} - \mathbf{z}(x)) \cdot \mathbf{u}(x).$$

The integral surfaces \tilde{S} of the distribution $\tilde{\mathbf{e}}$ are the intersections of the future light cones $l^+(z_i)$ with vertices on the world line $\mathcal{L}^{(1)}$ and those 3-planes $S^\perp(z_j)$ which are orthogonal to the particle velocities \mathbf{u}_i :

$$S_{ij} = l^+(z_i) \cap S^\perp(z_j) \quad (s_i \leq s_j).$$

The surfaces \tilde{S} can be considered as 2-dimensional wave surfaces propagating along the null direction field $\mathbf{n}(x)$. The set of all surfaces \tilde{S} , evolving from a single one when time goes on, fills the light cone $l^+(z)$. All surfaces \tilde{S} are 2-spheres within the Euclidean hyperplanes S^\perp with radius ϱ .

Once $\mathbf{z}(x)$ is defined, one knows the unit tangent vector \mathbf{u} to the world line in \mathbf{z}

$$\mathbf{u}(x) = \left. \frac{d\mathbf{z}}{ds} \right|_{s(x)} =: \dot{\mathbf{z}}. \quad (9)$$

As the second coordinate of x , one takes the "retarded distance" $\varrho(x)$

$$\varrho(x) = (\mathbf{x} - \mathbf{z}(x)) \cdot \mathbf{u}(x). \quad (10)$$

The fields $s(x)$ and $\varrho(x)$ over M_4^- refer to the \mathbb{R}^2 -factor of M_4^- . In order to find the two remaining coordinates (ϑ and φ , say) for the S^2 -factor of M_4^- , we have to map M_4^- onto the sphere S^2 . This can be done in the following way: Choose an orthonormal

triad of space-like vectors $\{\mathbf{e}_i, i=1,2,3\}$ at an arbitrary point \mathbf{z} of the world line $\mathcal{L}^{(1)}$ and shift this triad along the whole of $\mathcal{L}^{(1)}$ by Fermi-Walker transport

$$\begin{aligned} \dot{\mathbf{e}}_i &:= d\mathbf{e}_i/ds = \dot{\mathbf{u}}(\mathbf{u} \cdot \mathbf{e}_i) - \mathbf{u}(\dot{\mathbf{u}} \cdot \mathbf{e}_i), \\ \dot{\mathbf{u}} &:= d\mathbf{u}/ds, \quad (\mathbf{e}_i \cdot \mathbf{e}_j) = -\delta_{ij}, \quad (\mathbf{u} \cdot \mathbf{e}_i) \equiv 0. \end{aligned} \quad (11)$$

The image $(\vartheta, \varphi)(x)$ of $x \in M_4^-$ in S^2 can then be given by

$$\cos \vartheta(x) = \varrho^{-1}(\mathbf{x} - \mathbf{z}(x)) \cdot \mathbf{e}_3(s) =: (\mathbf{n} \cdot \mathbf{e}_3), \quad (12)$$

where we have introduced the light-like vector \mathbf{n} through

$$\mathbf{n}(x) = \varrho^{-1}(\mathbf{x} - \mathbf{z}(x)), \quad (\mathbf{n} \cdot \mathbf{n}) = 0. \quad (13)$$

Further, the angle $\varphi(x)$ is defined as

$$\tan \varphi(x) = (\mathbf{n} \cdot \mathbf{e}_1)^{-1}(\mathbf{n} \cdot \mathbf{e}_2). \quad (14)$$

Thus, an arbitrary point $x \in M_4^-$ is parametrized as

$$\mathbf{x} = x^\nu \mathbf{E}_\nu = \mathbf{z}(x) + \varrho \mathbf{n}(x), \quad (\mathbf{E}_\nu \equiv \partial_\nu). \quad (15)$$

The coordinate transformation $\{x^\nu\} \rightarrow \{s, \varrho, \vartheta, \varphi\}$ is accompanied by the transition of the coordinate basis \mathbf{E} of the tangent space $T_x(M_4^-)$ to the anholonomic basis $\mathbf{e}(x) := \{\mathbf{u}; \mathbf{v}, \mathbf{k}, \mathbf{h}\}$

$$\begin{aligned} \mathbf{v}(x) &:= \mathbf{n}(x) - \mathbf{u}(x), \quad \mathbf{k} := \partial \mathbf{v} / \partial \vartheta, \\ \mathbf{h} &:= (\sin \vartheta)^{-1} \partial \mathbf{v} / \partial \varphi. \end{aligned} \quad (16)$$

Denoting the cotangent vectors induced by the Minkowski metric $\{g_{\mu\nu}\}$ as $\mathbf{u}, \mathbf{n}, \mathbf{v}, \mathbf{k}, \mathbf{h}$ etc., we have the transformation formulae

$$\begin{aligned} \mathfrak{d}s &= \mathbf{n}, \\ \mathfrak{d}\varrho &= -\mathbf{v} + \varrho(\mathbf{n} \cdot \dot{\mathbf{u}})\mathbf{n}, \\ \mathfrak{d}\vartheta &= -\varrho^{-1}\mathbf{k} + (\dot{\mathbf{u}} \cdot \mathbf{k})\mathbf{n}, \\ \sin \vartheta \mathfrak{d}\varphi &= -\varrho^{-1}\mathbf{h} + (\dot{\mathbf{u}} \cdot \mathbf{h})\mathbf{n}, \end{aligned} \quad (17)$$

and the converse is

$$\begin{aligned} \mathfrak{d}\mathbf{x} &= \mathbf{u} \otimes \{[1 - \varrho(\mathbf{n} \cdot \dot{\mathbf{u}})]\mathfrak{d}s + \mathfrak{d}\varrho\} \\ &+ \mathbf{v} \otimes \{-\varrho(\mathbf{n} \cdot \dot{\mathbf{u}})\mathfrak{d}s + \mathfrak{d}\varrho\} \\ &+ \mathbf{k} \otimes \{\varrho \mathfrak{d}\vartheta - \varrho(\dot{\mathbf{u}} \cdot \mathbf{k})\mathfrak{d}s\} \\ &+ \mathbf{h} \otimes \{\varrho \sin \vartheta \mathfrak{d}\varphi - \varrho(\dot{\mathbf{u}} \cdot \mathbf{h})\mathfrak{d}s\}. \end{aligned} \quad (18)$$

II. Bundle Reduction

The manifold M_4^- , as defined in Sect. I, is clearly a four-dimensional submanifold of the flat space

M_4 and therefore its tangent bundle τ_4^- is trivial: $\tau_4^- = M_4^- \times \mathbb{R}^4$. Using the usual connection, which can always be defined in a canonical way for a flat base manifold, we look for a representative ω of this canonical connection in the associated bundle of orthogonal frames A_4^- . To this end, we use the frame $\mathbf{e}(x)$ defined in (16), and find for the canonical connection $\omega(x)$ through the application of the canonical covariant derivative $\nabla \equiv \partial$ to that frame $\mathbf{e}(x)$:

$$\begin{aligned}\partial \mathbf{e}(x) &= \mathbf{e}(x) \otimes \omega(x), \\ \omega(x) &= \mathfrak{B}(x) + \mathfrak{U}(x),\end{aligned}\quad (19)$$

where* ($a=1, 2, 3$)

$$\begin{aligned}\mathfrak{B} &= \mathfrak{B}_a t^a, \quad \mathfrak{U} = \mathfrak{U}_a T^a, \\ \mathfrak{B}_a &= -(\dot{\mathbf{u}} \cdot \mathbf{e}_a) \mathfrak{n}, \\ \mathfrak{U}^1 &= \cos \vartheta \, \mathfrak{d}\varphi = \cot \vartheta \cdot \mathfrak{U}^2, \\ \mathfrak{U}^2 &= -\sin \vartheta \, \mathfrak{d}\varphi = \varrho^{-1} \mathfrak{h} - (\dot{\mathbf{u}} \cdot \mathbf{h}) \mathfrak{n}, \\ \mathfrak{U}^3 &= \mathfrak{d}\vartheta = -\varrho^{-1} \mathfrak{k} + (\dot{\mathbf{u}} \cdot \mathbf{k}) \mathfrak{n}.\end{aligned}\quad (20)$$

Here, t^a and T^a are the space and time splittings of the Lorentz group generators, obeying the commutation relations

$$\begin{aligned}[T^a, t^b] &= \varepsilon^{ab}_c t^c, \\ [t^a, t^b] &= -\varepsilon^{ab}_c T^c, \\ [T^a, T^b] &= \varepsilon^{ab}_c T^c.\end{aligned}\quad (21)$$

Since we are working in a trivial bundle, there is no curvature

$$\Omega := \mathfrak{d}\omega + \omega \wedge \omega \equiv 0. \quad (22)$$

However, the reduced quantities become non-trivial. The connections $\hat{\omega}$ and $\tilde{\omega}$ due to the bundle reduction (6)

$$\hat{\omega} = \hat{P} \cdot \omega \cdot \hat{P} = \mathfrak{B}^1 \cdot \hat{t}^1, \quad (23)$$

$$\tilde{\omega} = \tilde{P} \cdot \omega \cdot \tilde{P} = \mathfrak{U}^1 \cdot \tilde{T}^1 \quad (24)$$

are obtained by the projection of the canonical connection ω onto the subalgebras $\mathcal{S}_\mathcal{O}(1,1)$ and $\mathcal{S}_\mathcal{O}(2)$ of the Lorentz algebra $\mathcal{S}_\mathcal{O}(1,3)$. The curvatures of these reduced connections are

$$\hat{\Omega} = \mathfrak{d}\hat{\omega} + \hat{\omega} \wedge \hat{\omega} = \mathfrak{d}\mathfrak{B}^1 \cdot \hat{t}^1, \quad (25)$$

$$\tilde{\Omega} = \mathfrak{d}\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = \mathfrak{d}\mathfrak{U}^1 \cdot \tilde{T}^1. \quad (26)$$

Using the connection coefficients found in (20) we see that the curvature field $\mathfrak{d}\mathfrak{B}^1$ in the $\text{SO}(1,1)$ bundle \hat{A}_4 is just the radiation part ${}^r\mathfrak{F}$ of the well-

known Lienard-Wiechert field (charge unity absorbed into the field)

$$\begin{aligned}{}^r\mathfrak{F} &:= \mathfrak{d}\mathfrak{B}^1 = -\mathfrak{d}[(\mathbf{n} \cdot \dot{\mathbf{u}}) \mathfrak{n}] \\ &= \varrho^{-1} \mathfrak{n} \wedge \{ \dot{\mathbf{u}} + (\mathbf{n} \cdot \dot{\mathbf{u}}) \mathfrak{v} \}.\end{aligned}\quad (27)$$

On the other hand, the curvature field $\mathfrak{d}\mathfrak{U}^1$ in the $\text{SO}(2)$ bundle is the dual of the Lienard-Wiechert field ${}^{\text{LW}}\mathfrak{F}$ itself

$$\begin{aligned}{}^*\mathfrak{F} &:= \mathfrak{d}\mathfrak{U}^1 = -\varrho^{-2} \mathfrak{k} \wedge \mathfrak{h} - \varrho^{-1} \mathfrak{n} \\ &\quad \wedge \{ (\dot{\mathbf{u}} \cdot \mathbf{h}) \mathfrak{k} - (\dot{\mathbf{u}} \cdot \mathbf{k}) \mathfrak{h} \},\end{aligned}\quad (28)$$

$$\begin{aligned}{}^{\text{LW}}\mathfrak{F} &= {}^*\mathfrak{F} = \varrho^{-2} \mathfrak{v} \wedge \mathfrak{u} + \varrho^{-1} \mathfrak{n} \\ &\quad \wedge \{ \mathfrak{u} + (\mathbf{n} \cdot \dot{\mathbf{u}}) \mathfrak{v} \}.\end{aligned}\quad (29)$$

Finally, it is a quite elementary matter to verify further that the Lienard-Wiechert field (29) belongs to one charge unit on the source travelling along the world line $\mathcal{L}^{(1)}$:

$$(4\pi)^{-1} \int_{C^{(2)}} {}^*\mathfrak{F} = -1 \quad (30)$$

i.e., we have $n = -1$ for the charge integral (2) over any 2-cycle $C^{(2)}$ surrounding the world line $\mathcal{L}^{(1)}$ [7].

Thus, we have achieved our main goal of this section, namely the identification of the geometric notion of curvature in an appropriate fibre bundle with the physical notion of Maxwell field of an arbitrarily accelerated point charge. Moreover, we have demonstrated the physical relevance of the curvature field in the $\text{SO}(1,1)$ bundle: It is just the radiation part of the Lienard-Wiechert field (29). This identification of geometric and physical quantities is a simple illustration of the Misner-Wheeler philosophy: ‘‘Classical physics as geometry’’ [8].

Comparing the two expressions for the bundle curvatures ${}^*\mathfrak{F}$ and ${}^r\mathfrak{F}$, one sees that the curvature ${}^*\mathfrak{F}$ in the $\text{SO}(2)$ bundle can always be attributed the property of being an electromagnetic field, provided Maxwell’s equations are satisfied ($\mathfrak{d}{}^*\mathfrak{F} = \delta{}^*\mathfrak{F} = 0$). On the other hand, since ${}^r\mathfrak{F}$ does not satisfy both halves of Maxwell’s equations ($\delta{}^r\mathfrak{F} \neq 0$), the interpretation of the $\text{SO}(1,1)$ curvature ${}^r\mathfrak{F}$ as a radiation field is dependent from the particular bundle construction used in this paper (for a short discussion of the generalizability of the present results see the last section of the paper).

As a final remark, let us point out that retarded potentials up to the third order can be found for the Lienard-Wiechert field (29) (Appendix).

III. Radiative Gauge Transformations

In this section, we turn to the peculiar fact that the proper gauge group for the radiation field $r\tilde{\gamma}$ is $SO(1,1)$ instead of $SO(2)$, which everyone assumes to be the right internal symmetry group of ordinary electrodynamics. According to this circumstance, we shall now consider (global) $SO(1,1)$ transformations of the (global) radiation potential ${}^1\mathfrak{B}$.

A change of frame

$$\begin{aligned}\hat{e}'(x) &= \hat{e}(x) \cdot \hat{A}(x), \quad \hat{A} \in SO(1,1) \\ (\hat{e}(x) &\equiv \{\mathbf{k}(x), \mathbf{h}(x)\})\end{aligned}\quad (31)$$

induces the change of the connection $\hat{\omega}$

$$\hat{\omega}' = a d_{\hat{A}^{-1}} \hat{\omega} + \hat{A}^{-1} \cdot \flat \hat{A}. \quad (32)$$

For the radiation potential, this means on account of (23) and (5), where \hat{A} is explicitly defined,

$${}^1\mathfrak{B}' = {}^1\mathfrak{B} - \flat\beta. \quad (33)$$

Now, choose for the gauge parameter β

$$\beta(x) = -\ln \varrho(x), \quad (34)$$

observe (17) and find

$${}^1\mathfrak{B}' = -\varrho^{-1} \flat. \quad (35)$$

It is easily verified by explicit exterior differentiation that the new potential ${}^1\mathfrak{B}'$ also yields the radiation field $r\tilde{\gamma}$ in (27).

On the other hand, the Lienard-Wiechert field has also a global potential

$${}^{LW}\tilde{\gamma} = \flat {}^{LW}\mathfrak{A}, \quad {}^{LW}\mathfrak{A} = \flat \varrho. \quad (36)$$

Hence, the bound field $\flat\tilde{\gamma} \equiv {}^{LW}\tilde{\gamma} - r\tilde{\gamma}$ exhibits the same property

$$\flat\tilde{\gamma} = \flat\flat\mathfrak{A}, \quad \flat\mathfrak{A} = \varrho^{-1} \cdot \flat. \quad (37)$$

In this way, one can have light-like potentials as well for the bound field $\flat\tilde{\gamma}$ as for its radiative counterpart $r\tilde{\gamma}$.

The radiative gauge transformation described in this section was recently discussed in non-covariant way in [9]. The space-like potential (35) was first found in [10].

IV. A further Geometric Approach to the Radiation Field

Based on the space-like radiation potential (35) one can give a further geometric characterization

of the radiation part $r\tilde{\gamma}$ of the Lienard-Wiechert field ${}^{LW}\tilde{\gamma}$. This observation is connected with the fact that the distributions $\hat{e} = \{\mathbf{u}, \mathbf{v}\}$, $\tilde{e} = \{\mathbf{k}, \mathbf{h}\}$ are involutive and hence are integrable.

Indeed, one computes the following commutators

$$\begin{aligned}[\mathbf{k}, \mathbf{h}] &= \cot \vartheta \cdot \varrho^{-1} \cdot \mathbf{h} \in \tilde{e}, \\ [\mathbf{u}, \mathbf{v}] &= -(\mathbf{n} \dot{\mathbf{u}}) \mathbf{n} \in \hat{e}, \\ [\mathbf{k}, \mathbf{n}] &= 0 \in \bar{e} \equiv \{\mathbf{k}, \mathbf{h}, \mathbf{n}\}, \\ [\mathbf{h}, \mathbf{n}] &= 0 \in \bar{e}.\end{aligned}\quad (38)$$

The Frobenius integrability theorem now tells us that we have three different integral manifolds. These are:

- i) two 2-dimensional hypersurfaces \hat{S} , \tilde{S} , the tangent spaces of which are built by \hat{e} and \tilde{e} , resp.
- ii) the three-dimensional forward light cone $\tilde{S} \equiv l^+(z)$ with vertex z on the world line $\mathcal{L}^{(1)}$ (see Figure 1). The tangent space of the latter is formed by the distribution $\bar{e} = \{\mathbf{k}, \mathbf{h}, \mathbf{n}\}$.

Before further proceeding in the study of the Lienard-Wiechert field, we want to introduce some geometric notions (cf. [11]).

Denoting the 1-form basis as

$$\tilde{\mathcal{E}} := \begin{pmatrix} -\mathfrak{k} \\ -\mathfrak{h} \end{pmatrix}, \quad \hat{\mathcal{E}} := \begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \quad (39)$$

we introduce the “torsion 1-forms” Σ by application of the covariant derivatives $\tilde{\mathcal{D}}$, $\hat{\mathcal{D}}$ in the corresponding cotangent bundles

$$\tilde{\mathcal{D}}\tilde{\mathcal{E}} = \flat\tilde{\mathcal{E}} + \tilde{\omega} \wedge \tilde{\mathcal{E}} = -\tilde{\Sigma} \wedge \tilde{\mathcal{E}}, \quad (40)$$

$$\hat{\mathcal{D}}\hat{\mathcal{E}} = \flat\hat{\mathcal{E}} + \hat{\omega} \wedge \hat{\mathcal{E}} = -\hat{\Sigma} \wedge \hat{\mathcal{E}}. \quad (41)$$

The torsion one-forms Σ do not belong to the intrinsic geometry of the corresponding fibre bundles but rather describe the special way how the reduced bundles are embedded in the trivial bundle (\mathcal{A}_4^-) . The most striking property of the torsion one-forms referring to their extrinsic nature is the fact that they annihilate the corresponding distributions, i.e. $\tilde{\Sigma}$ annihilates \tilde{e} and $\hat{\Sigma}$ annihilates \hat{e} . Due to this property, the one-form matrices Σ can be decomposed as

$$\tilde{\Sigma} = -\mathcal{S}_a \hat{\mathcal{E}}^a, \quad \hat{\Sigma} = -\mathcal{S}_i \tilde{\mathcal{E}}^i. \quad (42)$$

(Latin indices i, j, k, \dots are used to count vectors

and forms in $\hat{\mathbf{e}}$ and similarly a, b, c, \dots refer to the normal distribution $\hat{\mathbf{e}}$.

An $\text{SO}(2)$ transformation (3) transforms the \mathcal{S} -matrices as

$$\mathcal{S}_{a'} = a d_{\tilde{A}-1} \mathcal{S}_a, \quad \tilde{A} \in \text{SO}(2),$$

whereas an $\text{SO}(1,1)$ transformation \hat{A} transforms the \mathcal{S} -matrices as a vector quantity

$$\mathcal{S}_{a'} = \hat{A}^b_a \mathcal{S}_b.$$

Similar statements hold for the matrices \mathcal{S}_i . The geometric meaning of the \mathcal{S} matrices is the following: Given a normal section $\hat{\mathbf{w}} \in \hat{\mathbf{e}}$ with components w^a relative to $\hat{\mathbf{e}}(x)$ and a section $\tilde{l} \in \tilde{\mathbf{e}}$ with components \mathfrak{l} relative to $\tilde{\mathbf{e}}(x)$ then the projection of the covariant derivative of $\hat{\mathbf{w}}$ onto $\tilde{\mathbf{e}}$ is

$$\tilde{\mathfrak{D}} \cdot \nabla_{\tilde{l}} \hat{\mathbf{w}} = w^a \mathfrak{l}^i \mathcal{S}^i_{aj} \tilde{\mathbf{e}}_i. \quad (43)$$

The projectors involved here are

$$\begin{aligned} \tilde{\mathfrak{P}} &= -\mathbf{k} \otimes \mathfrak{k} - \mathbf{h} \otimes \mathfrak{h}, \\ \hat{\mathfrak{P}} &= \mathbf{u} \otimes \mathfrak{u} - \mathbf{v} \otimes \mathfrak{v}. \end{aligned} \quad (44)$$

This means the matrix $\mathcal{S}_w := w^a \mathcal{S}_a$ describes that endomorphism in $\hat{\mathbf{e}}$ which maps the directional vector \tilde{l} onto the projection of the covariant derivative of $\hat{\mathbf{w}}$. If \mathcal{S}_w is proportional to the identity $\tilde{I} : \mathcal{S}_w \sim \tilde{I}$, we speak of an umbilical normal section $\hat{\mathbf{w}}$ of the surface \tilde{S} .

An important geometric quantity derivable from the torsion 1-form Σ is the "mean curvature 1-form" σ :

$$\sigma = \frac{1}{2} \text{tr } \Sigma. \quad (45)$$

For the surface \tilde{S} , the torsion 1-form is calculated as

$$\tilde{\Sigma} = -\varrho^{-1} \mathfrak{v} \cdot \tilde{I}, \quad \tilde{\sigma} = -\varrho^{-1} \mathfrak{v} \quad (46)$$

which says that $\mathfrak{v}(x)$ is an umbilical normal section of $\tilde{S}(\mathcal{S}_{\mathfrak{v}} = \varrho^{-1} \cdot \tilde{I})$. Now, take the covariant derivative and the trace of (46), observe the potential (35) for the radiation field (27) and find

$$\mathfrak{r}\tilde{\mathfrak{D}} = \frac{1}{2} \text{tr}(\tilde{\mathfrak{D}} \tilde{\Sigma}) = \mathfrak{v} \tilde{\sigma}. \quad (47)$$

This equation says that the radiation part $\mathfrak{r}\tilde{\mathfrak{D}}$ of the Lienard-Wiechert field ${}^{\text{LW}}\tilde{\mathfrak{D}}$ is just the derivative of the mean curvature 1-form $\tilde{\sigma}$ of those 2-

surfaces \tilde{S} , the tangent planes of which were used to construct the original fibre bundle with the dual of the Lienard-Wiechert field as curvature. Besides the identification of the radiation part $\mathfrak{r}\tilde{\mathfrak{D}}$ with the curvature in the normal bundle, this is a second purely geometric characterization of the radiation field (based on the extrinsic geometry of the $\text{SO}(2)$ bundle!)

V. Discussion

The foregoing considerations have shown that a geometric structure can be cast over Minkowski space such that the relevant geometric objects of this structure coincide with the field theoretic quantities used by physicists for a long time to describe the electromagnetic field of a single point charge. Now there arise a lot of questions concerning the generalizability of the present calculations.

Remaining for a moment at the one-particle case, one first wants to know whether a distortion of the distribution $\tilde{\mathbf{e}}$, leaving the curvature ${}^*\tilde{\mathfrak{D}}$ invariant, leaves also invariant the curvature $\mathfrak{r}\tilde{\mathfrak{D}}$ in the normal bundle, or in other words, does the curvature $\mathfrak{r}\tilde{\mathfrak{D}}$ react more sensitive to a change of the 2-planes than does ${}^*\tilde{\mathfrak{D}}$? Secondly, one learns from Sect. V, that the integrability of the distribution $\tilde{\mathbf{e}}$ is necessary for the second geometric characterization of the radiation field. Therefore, the question arises whether the integral surfaces \tilde{S} can be deformed in such a way that their mean curvature 1-form remains a potential for the radiation field $\mathfrak{r}\tilde{\mathfrak{D}}$?

From the physical point of view, the more interesting questions are those connected with the generalization to the many particle case. Is it possible to describe the retarded particle field of N point charges also by the intrinsic curvature of a 2-plane bundle? Is the curvature in the normal bundle still identical to the (local) radiation field generated by the N particles? Can the distributions still be chosen to be integrable, or expressed in physical terms, do there exist 2-dimensional wave surfaces \tilde{S} propagating along null directions \mathbf{n} ? Exists there also a 3-dimensional integrable distribution, namely the generalization of the future light cone $l^+(z)$, which is swept out by the 2-dimensional wave surfaces \tilde{S} when time goes on?

We intend to present answers to some of these questions in a future paper.

Acknowledgement

The author is indebted to Prof. J. M. Leinaas for valuable comments resulting in an improved version of the paper.

Appendix

Retarded Potentials of Higher Order for the Lienard-Wiechert Field

The Lienard-Wiechert potential (36) satisfies the Lorentz gauge condition

$$\delta \text{LW}\mathfrak{U} = 0. \quad (\text{A.1})$$

Therefore, a potential \mathfrak{C} exists such that

$$\text{LW}\mathfrak{U} = -\delta \mathfrak{C} \quad (\text{A.2})$$

on account of the identity $\delta\delta = 0$.

One easily finds for the 2-form \mathfrak{C} (up to an additive exact codifferential)

$$\mathfrak{C} = \tfrac{1}{2} \mathfrak{u} \wedge \mathfrak{n}. \quad (\text{A.3})$$

which has a global meaning in M_4^- . Further, one computes

$$\delta \mathfrak{C} = 0, \quad (\text{A.4})$$

and hence we must have (up to an additive exact differential)

$$\mathfrak{C} = \delta \mathfrak{N} \quad (\text{A.5})$$

because of $\delta\delta = 0$. The 1-form \mathfrak{N} is found to be

$$\mathfrak{N} = \tfrac{1}{2} \varrho \mathfrak{n}. \quad (\text{A.6})$$

The Lienard-Wiechert field (29) can now be written as

$$\text{LW}\mathfrak{F} = \delta \text{LW}\mathfrak{U} = -\delta \delta \mathfrak{C}. \quad (\text{A.7})$$

But because of (A.4) we can write this also as

$$\text{LW}\mathfrak{F} = -(\delta\delta + \delta\delta)\mathfrak{C} \equiv -\square \mathfrak{C}, \quad (\text{A.8})$$

where \square is the d'Alembertian (Laplacean in Minkowski space). Using once more (A.6), $\text{LW}\mathfrak{F}$ can be cast into the shape

$$\text{LW}\mathfrak{F} = -\delta \square \mathfrak{N} = -\square \delta \mathfrak{N}. \quad (\text{A.9})$$

In this way, the 1-form \mathfrak{N} is a potential of the third order for the Lienard-Wiechert field. Observe that all these potentials of higher order are strictly retarded in the same sense as is $\text{LW}\mathfrak{F}$ itself.

Now take the dual of Eq. (A.8) and find for the curvature field $*\mathfrak{F}$ of the $\text{SO}(2)$ bundle

$$*\mathfrak{F} = \square * \mathfrak{C} = \tfrac{1}{2} \square (\mathfrak{f} \wedge \mathfrak{h}), \quad (\text{A.10})$$

i.e., also this field is the Laplacean of a strictly retarded potential.

- [1] J. M. Leinaas, Fortschr. Phys. **28**, 579 (1980).
- [2] B. Felsager and J. M. Leinaas, Nucl. Phys. **B166**, 162 (1980).
- [3] F. Gliozzi, Nucl. Phys. **B141**, 379 (1978).
- [4] Kobayashi and Nomizu, Foundations of Differential Geometry, Interscience Publishers 1969, Vol. s I, II.
- [5] C. Teitelboim, D. Villarroel, and C. G. van Weert, Riv. Nuovo Cim. **3**, no. 9 (1980).
- [6] B. Felsager and J. M. Leinaas, Ann. Phys. **130**, 461 (1980).
- [7] The general charge integral (2) over the field (1) assumes integer values according to the property of the

- field (1) of being an element of the cohomology group $H^2(M_4^-, \mathbb{Z})$.
- [8] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957).
- [9] K. H. Yang, Phys. Lett. **81A**, 125 (1981).
- [10] C. Teitelboim, Phys. Rev. **D3**, 297 (1971).
- [11] B. Chen, Geometry of Submanifolds, Marcel Dekker Inc., New York 1973.

* Latin indices are raised and lowered by the Euclidean metric $\{\eta_{ab}\} = \{\delta_{ab}\} = \text{diag } (1, 1, 1)$, i. e. $\mathfrak{U}^2 = \mathfrak{U}_2$, $\varepsilon_{123} = \varepsilon^{123} = \varepsilon^{12\bar{3}} = 1$, etc.